Optimisation Discrete

Linear Programming: Simplex Method

Sonia Cafieri

ENAC

sonia.cafieri@enac.fr

Outline

Linear Programming : formulation

Linear Programming problem in *standard form*:

$$(LP) \begin{cases} \min & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & \sum a_{ij} x_j \le b_i \quad i = 1, \ldots, m, \; m < n \\ & x_j \ge 0 \qquad\qquad j + 1, \ldots, n \end{cases}$$

where $x \in \mathbb{R}^n$ are the decision variables, $c \in \mathbb{R}^n$ is a cost vector, A is a $m \times n$ matrix (rang m) and $b \in R^m$.

Linear Programming : geometry

Definition

A set $\{A_i | i \in \beta\}$ of m linearly independent columns of A is a basis of A.

The variables $\{x_i | i \in \beta\}$ corresponding to the indices β of the basis are called basic variables. All other variables are called nonbasic variables.

Let $(B|N)$ be a partition into basic and nonbasic columns of A.
Let (x_B, x_N) the corresponding partition of the variables.

Definition

Given a polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \ge 0\}$, the feasible vectors x having $x_B = B^{-1}b \ge 0$ and $x_N = 0$ are called basic feasible solutions (bfs) of P .

Linear Programming : geometry

Lemma

Given a polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ *and a bfs* x^* *for* P*, there exists a cost vector* $c \in \mathbb{R}^n$ *such that* x^* *is the unique optimal solution of the problem* $\min\{c^T x | x \in P\}$.

bfs solutions correspond to vertices of P:

Theorem

Given a polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$*, any bfs for* P *is vertex of* P*, and vice-versa.*

Linear Programming: geometry

Theorem

For any feasible bounded linear program there exists an optimal solution that corresponds to a **vertex of the polyhedron** *that describes the set of feasible solutions.*

Note: since there are finitely many vertices in a polyhedron in $\mathbb{R}^n$,
the (continuous) LP problem can be transformed into a finite search.

The Simplex Method

Basic observations

The Simplex Method is based on the previous theorem: it proceeds by hopping from a vertex of the polyhedron to another in such a way that the objective function never increases.

Basic observations:

- To verify whether a vertex of a polyhedron is a local minimum w.r.t. a linear form it suffices to check that all adjacent vertices have higher associated objective function value.
- Polyhedra are convex sets and linear forms are convex functions, so any local minimum is also a global minimum.

The Simplex Method

The Simplex Method is "a systematic procedure for generating and testing candidate vertex solutions to a linear program." (Gill, Murray, and Wright)

- It begins at an arbitrary corner of the solution set.
- At each iteration, selects the variable that will produce the largest change towards the minimum solution.
- That variable replaces one of the adjacents that is most severely restricting it, thus moving the Simplex Method to a different corner of the solution set (and closer to the final solution).

The Simplex Method

The Simplex Method is "a systematic procedure for generating and testing candidate vertex solutions to a linear program." (Gill, Murray, and Wright).

Note:

- The Simplex Method can determine if no solution actually exists.
- The algorithm is greedy since it selects the best choice at each iteration without needing information from previous or future iterations.

The Simplex Method

At each step, the same process is repeated with a new variable becoming basic as another becomes nonbasic.

Eventually, one of three things will happen:

- a solution may occur where no nonbasic variable will decrease the cost, in which case the current solution is the *optimal solution*.
- a non-basic variable might increase to infinity without causing a basic-variable to become zero, resulting in an *unbounded solution*.
- *no solution* may actually exist and the Simplex Method must abort.

The Simplex Method : an Example (1/9)

$$
\begin{cases}
\max & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 \leq 5 \\
& 4x_1 + x_2 + 2x_3 \leq 11 \\
& 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& x_1, x_2, x_3 \geq 0
\end{cases}
$$

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[1](See [Chvatal,1983] and `http://classes.apl.washington.edu/Math407Summer2005/dictionary2.1.htm`)

The Simplex Method : an Example (2/9)

Rewrite with slack variables:

$$
\begin{cases}
\max & z = 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} & s_1 = 5 - 2x_1 - 3x_2 - x_3 \\
& s_2 = 11 - 4x_1 - x_2 - 2x_3 \\
& s_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{cases}
$$

This layout is called a *dictionary*.

(so, dictionaries contain a representation of the set of equations appropriately adjusted to the current basis)

- Variables on the left are the *basic variables*.
- Variables on the right are the *nonbasic variables*.

The Simplex Method : an Example (3/9)

$$\begin{cases} \max & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & s_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & s_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & s_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The basic feasible solution sets the variables on the right side of the equations to zero and then solves for the ones on the left side.

For our example, the basic feasible solution is:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ s_1 = 5,\ s_2 = 11,\ s_3 = 8,\ z = 0$$

This solution is also called a *dictionary solution*.

The Simplex Method : an Example (4/9)

$$\begin{cases} \max & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & s_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & s_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & s_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

We notice that:

- If we increase x_1 and keep $x_2 = x_3 = 0$, then z (the objective function) increases (because the coefficient of x_1 in the equation for z is positive)
- How much can we increase x_1? Until s_1 goes to zero.
- Do it. The result is that $x_1 \geq 0$ and $s_1 = 0$.
  That is, x_1 becomes *basic* and s_1 becomes *non basic*.
- This step is called a *pivot*.

The Simplex Method : an Example (5/9)

(First) Pivot : exchange the role of two variables $x_1 \leftrightarrow s_1$

We solve for the variable x_1 in the row corresponding to s_1:

$$x_1 = 2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3$$

and replace the row corresponding to s_1 with this equation and we use this equation to replace all other occurrences of x_1 in the problem:

$$\begin{cases} \max & z = 5(2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3) + 4x_2 + 3x_3 \\ \text{s.t.} & x_1 = 2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3 \\ & s_2 = 11 - 4(2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3) - x_2 - 2x_3 \\ & s_3 = 8 - 3(2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3) - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The Simplex Method : an Example (6/9)

(First) Pivot : exchange the role of two variables $x_1 \leftrightarrow s_1$

Regruping terms:

$$\begin{cases} \max & z = 12.5 - 2.5s_1 - 3.5x_2 + 0.5x_3 \\ \text{s.t.} & x_1 = 2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3 \\ & s_2 = 1 + 2s_1 + 5x_2 \\ & s_3 = 0.5 + 1.5s_1 + 0.5x_2 - 0.5x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The bfs is:

$$x_1 = 2.5,\ x_2 = 0,\ x_3 = 0,\ s_1 = 0,\ s_2 = 1,\ s_3 = 0.5,\ z = 12.5$$

so, the value of z has increased from the previous problem representation.

The Simplex Method : an Example (7/9)

Second pivot.

$$\begin{cases} \max & z = 12.5 - 2.5s_1 - 3.5x_2 + 0.5x_3 \\ \text{s.t.} & x_1 = 2.5 - 0.5s_1 - 1.5x_2 - 0.5x_3 \\ & s_2 = 1 + 2s_1 + 5x_2 \\ & s_3 = 0.5 + 1.5s_1 + 0.5x_2 - 0.5x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

We notice that:

- If we increase x_3 and keep $x_2 = s_1 = 0$, then z (the objective function) increases (because the coefficient of x_3 in the equation for z is positive)
- How much can we increase x_3? Until s_3 goes to zero.
- Do it. The result is that $x_3 \geq 0$ and $s_3 = 0$.
  That is, x_3 becomes *basic* and s_3 becomes *non basic*.

The Simplex Method : an Example (8/9)

(Second) Pivot : $x_3 \leftrightarrow s_3$

We solve for the variable x_3 in the row corresponding to s_3:

$$x_3 = 1 + 3s_1 + x_2 - 2s_3$$

and replace the row corresponding to s_3 with this equation and we use this equation to replace all other occurrences of x_3 in the problem:

$$\begin{cases} \max & z = 12.5 - 2.5s_1 - 3.5x_2 + 0.5(1 + 3s_1 + x_2 - 2s_3) \\ \text{s.t.} & x_1 = 2.5 - 0.5s_1 - 1.5x_2 - 0.5(1 + 3s_1 + x_2 - 2s_3) \\ & s_2 = 1 + 2s_1 + 5x_2 \\ & x_3 = 1 + 3s_1 + x_2 - 2s_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The Simplex Method : an Example (9/9)

(Second) Pivot : $x_3 \leftrightarrow s_3$

Regrouping terms:

$$\begin{cases} \max & z = 13 - s_1 - 3x_2 - s_3 \\ \text{s.t.} & x_1 = 2 - 2s_1 - 2x_2 + s_3 \\ & s_2 = 1 + 2s_1 + 5x_2 \\ & x_3 = 1 + 3s_1 + x_2 - 2s_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The bfs is:

$$x_1 = 2,\ x_2 = 0,\ x_3 = 1,\ s_1 = 0,\ s_2 = 1,\ s_3 = 0,\ z = 13$$

so, the value of z has increased from the previous problem representation.

This is the optimal solution, because there is no feasible direction in which the objective function increases.

The Simplex Method

Note :

At each step : which non-basic variable should replace a basic variable?

The non-basic variable to select should be such that the value of the objective function does not decrease. The objective function is stated in terms of non-basic variables only, so its value does not decrease by selecting a non-basic variable whose coefficient in the objective function is positive.

Exercise

$$\begin{cases} \max & 400x_1 + 200x_2 \\ \text{s.t.} & 30x_1 + 20x_2 \leq 6000 \\ & 40x_1 + 10x_2 \leq 4000 \\ & x_1, x_2 \geq 0 \end{cases}$$

Solve the problem by the Simplex Method with Dictionaries.

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